The Wave Equations of the Electron.

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1. In a recent paper Dirac* has brilliantly removed the defects before existing in the mechanics of the electron, and has shown how the phenomena usually called the " spinning electron " fit into place in the complete theory. He applies to he problem the method of q-numbers and, using non-commutative algebra, exhibits the properties of a free electron, and of an electron in a central field of electric force. In a second paper† he also discusses the rules of combination and the Zeeman effect. There are probably readers who will share the present writer's feeling that the methods of non-commutative algebra are harder to follow, and certainly much more difficult to invent, than are operations of types long familiar to analysis. Wherever it is possible to do so, it is surely better to present the theory in a mathematical form that dates from the time of Laplace and Legendre, if only because the details of the calculus have been so much more thoroughly explored. So the object of the present work is to take Dirac's system and treat it by the ordinary methods of wave calculus. The chief

* ' Roy. Soc. Proc.,' A, vol. 117, p. 610 (1928).
† ' Roy. Soc. Proc.,' A, vol. 118, p. 351 (1928).

point of interest is perhaps the solution of the problem of the central field, which can be carried out exactly and leads to Sommerfeld's original formula for the hydrogen levels. But it is also of some interest to exhibit the relationship of the new theory to the previous equations which were derived empirically by the present writer.* It appears that those equations were an approximation to the new ones, derived by an approximate elimination of two of Dirac's four wave functions. We shall also review a few other points connected with the free electron, the emission of radiation from an atom and its magnetic moment, and shall outline a discussion of the Zeeman effect.

2. Dirac's guiding principle is that the "Hamiltonian equation" must be linear, and he adopts the form

$$p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \alpha_4 mc = 0,$$

where

$$\left.\begin{aligned} p_0 &= -\frac{h}{2\pi i}\frac{1}{c}\frac{\partial}{\partial t} + \frac{e}{c}\mathrm{V} \\ p_1 &= \frac{h}{2\pi i}\frac{\partial}{\partial x} + \frac{e}{c}\mathrm{A}_1, \text{ etc.} \end{aligned}\right\}, \tag{2.1}$$

V and A being scalar and vector potentials; while $\alpha_1 \ldots \alpha_4$ are four four-rowed matrices obeying the rules

$$\alpha_s^2 = 1, \qquad \alpha_s\alpha_t + \alpha_t\alpha_s = 0.$$

The α's are capable of an indefinite number of forms, and he gives rules for forming one set (though he does not write them out). The four-rowed matrices imply four wave functions which satisfy the simultaneous equations

$$\left.\begin{aligned} (p_0 + mc)\,\psi_1 + (p_1 - ip_2)\,\psi_4 + p_3\psi_3 &= 0 \\ (p_0 + mc)\,\psi_2 + (p_1 + ip_2)\,\psi_3 - p_3\psi_4 &= 0 \\ (p_0 - mc)\,\psi_3 + (p_1 - ip_2)\,\psi_2 + p_3\psi_1 &= 0 \\ (p_0 - mc)\,\psi_4 + (p_1 + ip_2)\,\psi_1 - p_3\psi_2 &= 0 \end{aligned}\right\}. \tag{2.2}$$

We shall take these, then, as our fundamental equations and discuss their solution, employing only the ordinary methods of differential equations.

The equations are very unsymmetrical, and it is, of course, necessary first to show that they can be restored to their original form when axes are changed or a relativity transformation is applied. The general formulæ are complicated (being best expressed by four-dimensional Cayley parameters), but it is sufficient to verify the result for certain simpler transformations which can be imagined

* 'Nature,' vol. 119, p. 282 (1927); 'Roy. Soc. Proc.,' A, vol. 116, p. 227 (1927).

applied successively. This is so straightforward that we need merely give the results.

(1) Relativity transformation

$$x = x', \qquad y = y',$$
$$z = z' \cosh \beta + ct' \sinh \beta,$$
$$ct = ct' \cosh \beta + z' \sinh \beta.$$

The equations are restored to their original form by

$$\psi_1' = \psi_1 \cosh \frac{\beta}{2} + \psi_3 \sinh \frac{\beta}{2},$$

$$\psi_2' = \psi_2 \cosh \frac{\beta}{2} - \psi_4 \sinh \frac{\beta}{2},$$

$$\psi_3' = \psi_3 \cosh \frac{\beta}{2} + \psi_1 \sinh \frac{\beta}{2},$$

$$\psi_4' = \psi_4 \cosh \frac{\beta}{2} - \psi_2 \sinh \frac{\beta}{2}.$$

(2) Rotation about z.

$$x = x' \cos \alpha - y' \sin \alpha,$$
$$y = y' \cos \alpha + x' \sin \alpha,$$
$$z = z', \qquad t = t'.$$

Then

$$\psi_1' = \psi_1 e^{i\alpha/2}, \qquad \psi_2' = \psi_2 e^{-i\alpha/2}, \qquad \psi_3' = \psi_3 e^{i\alpha/2}, \qquad \psi_4' = \psi_4 e^{-i\alpha/2}.$$

(3) Rotation about y.

$$z = z' \cos \alpha - x' \sin \alpha,$$
$$x = x' \cos \alpha + z' \sin \alpha,$$
$$y = y', \qquad t = t',$$

$$\psi_1' = \psi_1 \cos \frac{\alpha}{2} + \psi_2 \sin \frac{\alpha}{2},$$

$$\psi_2' = \psi_2 \cos \frac{\alpha}{2} - \psi_1 \sin \frac{\alpha}{2},$$

$$\psi_3' = \psi_3 \cos \frac{\alpha}{2} + \psi_4 \sin \frac{\alpha}{2},$$

$$\psi_4' = \psi_4 \cos \frac{\alpha}{2} - \psi_3 \sin \frac{\alpha}{2}.$$

These three transformations can build a group which represents any relativity transformation, and so the invariance is proved.

It is of some interest to consider this invariance a little further. The whole theory of general relativity is based on the idea of invariance of form, and here we have a system invariant in fact but not in form. Should it not be possible to give it formal invariance as well, and would not that be the right way to express our equations? It is so possible, but it is not hard to show that it requires no less than 16 quantities to do it,* viz., two scalars, two four-vectors and one six-vector, and even so each will have a real and imaginary part, so that we may say that 32 quantities are required! It seems quite preposterous to think that a single electron should require 32 equations to express its behaviour, and, moreover, these 32 will involve a large number of arbitrary inter-relations of no influence on the four quantities which are actually sufficient to describe it. Now the relativity theory is based on nothing but the idea of invariance, and develops from it the conception of tensors as a matter of necessity; and it is rather disconcerting to find that apparently something has slipped through the net,† so that physical quantities exist which it would be, to say the least, very artificial and inconvenient to express as tensors. It does not seem possible to make anything further out of the matter until it has developed more, and we shall be content with one observation. Unlike the electromagnetic equations, our wave equations are homogeneous, so that there is no external quantity, like the electric current, etc., which could, so to speak, anchor them down in form to a definite set of directions. Now, there ought to be something of the kind because of the electromagnetic field of the electron, which in classical theory is made responsible for its mass. So we may perhaps conclude that it is not to be expected that our equations will attain a final form until the terms in mc are eliminated, that is, until we know how to do in the quantum theory a calculation like that which gives electromagnetic mass in the classical.

In my earlier paper a similar question arose and was much more easily resolved. In that work there were only two functions instead of the four here, and it was an easy matter to throw them into space-vector form, though it involved having four equations instead of two with a corresponding arbitrariness

* We can express the equations as a group of 16 in

$$\alpha\psi_1 + \beta\psi_2 + \gamma\psi_3 + \delta\psi_4, \qquad \alpha\psi_4 + \beta\psi_3 + \gamma\psi_2 + \delta\psi_1, \text{ etc.},$$

with $\alpha\beta\gamma\delta$ arbitrary constants and can throw these into tensor form.

† Our equations (2.2) do not, of course, include gravitation, and this may be the hole in the net. But if gravitation were included, we should presumably be forced to introduce the tensor form, involving 16 complex or 32 real quantities, and this does not seem physically very plausible.

in the solution. It appeared reasonable to make the step from two to four, and so to gain the advantage of vector notation, but to expand from four to sixteen is a different matter, and suggests that even in the simpler case the expansion is rather artificial. Nevertheless, it is not without interest to exhibit Dirac's equations in the form of space-vectors, without bringing in the time as part of the vectorial system. This can be done by a method similar to the substitution (5.1) of my paper. Take two vectors X, Y and two scalars X_0, Y_0, and write p_1, p_2, p_3 as a vector p. The equations

$$\left.\begin{aligned} (p_0 + mc)\,Y_0 &= (p, X) \\ (p_0 + mc)\,Y - p\,.\,X_0 &= [p, X] \\ (p_0 - mc)\,X_0 &= (p, Y) \\ (p_0 - mc)\,X - p\,.\,Y_0 &= -[p, Y] \end{aligned}\right\} \tag{2.3}$$

can be combined together in pairs according to either of the following alternative schemes so as to give Dirac's equations:—

$$\left.\begin{aligned} \psi_1 &= Y_3 - iY_0 \\ \psi_2 &= Y_1 + iY_2 \\ \psi_3 &= iX_3 - X_0 \\ \psi_4 &= iX_1 - X_2 \end{aligned}\right\}, \qquad \left.\begin{aligned} \psi_1 &= -Y_1 + iY_2 \\ \psi_2 &= Y_3 + iY_0 \\ \psi_3 &= -iX_1 - X_2 \\ \psi_4 &= iX_3 + X_0 \end{aligned}\right\}.$$

So we might regard (2.3) as the primitive equations giving Dirac's twice over. Now the operation $p_1 = \frac{h}{2\pi i}\cdot\frac{\partial}{\partial x} + \frac{e}{c}A_1$ may be likened to a "covariant differentiation" (say, by the introduction of a fifth dimension in the manner of Klein*), and in this sense $[p, X]$ may be called *curl* X and (p, X), *div* X. In the same sense p_0 is a time differentiation, and we see that (2.3) bear a rather striking resemblance to the electromagnetic equations with X and Y for E and H, and X_0, Y_0 playing a role akin to electric and magnetic density. It does not seem possible to push this rather loose analogy farther at present, and again we have a hint as to the reason, because there is no electromagnetic analogue to the terms in mc; and this will only be supplied when we know how to calculate electromagnetic mass in the quantum theory.

3. The equations (2.2) are sufficient to determine the levels of any system; but that is not enough, for we also require to know the rules of combination, and for this we must have the extension of the "electric density" of Schrödinger's theory to the present case. In order to find the radiation of an atom

* Klein, 'Z. f. Physik,' vol. 37, p. 895 (1926).

we combine atom and radiation field into a single system, expressing the equations all together most conveniently by means of a variational principle.

The variational function is easy to construct. Multiply the equations (2.2) by $\delta\psi_1^*$, $\delta\psi_2^*$, $\delta\psi_3^*$, $\delta\psi_4^*$ respectively and add them together. Add on their conjugates and integrate over all space and time. Integrating by parts, this can be written as the variation of a single function. To this we must add $\frac{1}{8\pi}(\mathrm{H}^2 - \mathrm{E}^2)$, integrated over space and time, in order to give the dynamical effect of the radiation. Then, if we express H and E in terms of the potentials, and vary V and the A's as well as the ψ's and ψ^*'s, we shall obtain equations for V and the A's as well as (2.2) and their conjugates, and shall thus have the radiation completely linked with the material motion. In carrying out this process one point is to be noticed. The equations (2.2) are homogeneous and so do not fix the magnitudes of the ψ's. The equation for V is not so, but determines V as proportional to a quadratic expression in the ψ's. We naturally adjust this by normalising the ψ's so that the total charge shall be $-e$, the charge of an electron. It is more convenient to normalise the ψ's to unity, and therefore the terms $\frac{1}{8\pi}(\mathrm{H}^2 - \mathrm{E}^2)$ must be multiplied by a suitable constant. To save writing out many formulæ twice, we shall anticipate the knowledge of this constant; it is $-1/c$. Proceeding in this way we arrive at the variational function

$$
\begin{aligned}
\mathrm{S} = \iiiint dx\,dy\,dz\,dt \Big\{ & \frac{1}{2}\frac{h}{2\pi i}\Big[\psi_1^*\Big(-\frac{1}{c}\frac{\partial\psi_1}{\partial t} + \Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)\psi_4 + \frac{\partial}{\partial z}\psi_3\Big) \\
& + \psi_2^*\Big(-\frac{1}{c}\frac{\partial\psi_2}{\partial t} + \Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)\psi_3 - \frac{\partial}{\partial z}\psi_4\Big) \\
& + \psi_3^*\Big(-\frac{1}{c}\frac{\partial\psi_3}{\partial t} + \Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)\psi_2 + \frac{\partial}{\partial z}\psi_1\Big) \\
& + \psi_4^*\Big(-\frac{1}{c}\frac{\partial\psi_4}{\partial t} + \Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)\psi_1 - \frac{\partial}{\partial z}\psi_2\Big)\Big] + \text{the conjugate of the preceding} \\
& + mc(\psi_1\psi_1^* + \psi_2\psi_2^* - \psi_3\psi_3^* - \psi_4\psi_4^*) + \frac{e\mathrm{V}}{c}\sum_1^4 \psi_\lambda\psi_\lambda^* \\
& + \frac{e\mathrm{A}_1}{c}(\psi_1^*\psi_4 + \psi_2^*\psi_3 + \psi_3^*\psi_2 + \psi_4^*\psi_1) \\
& + \frac{e\mathrm{A}_2}{c}(-i\psi_1^*\psi_4 + i\psi_2^*\psi_3 - i\psi_3^*\psi_2 + i\psi_4^*\psi_1) \\
& + \frac{e\mathrm{A}_3}{c}(\psi_1^*\psi_3 - \psi_2^*\psi_4 + \psi_3^*\psi_1 - \psi_4^*\psi_2) \\
& - \frac{1}{8\pi c}\Big[(\text{curl A})^2 - \Big(\frac{1}{c}\frac{\partial \mathrm{A}}{\partial t} + \text{grad V}\Big)^2\Big]\Big\}. \qquad (3.1)
\end{aligned}
$$

If we vary this function it is easy to see that we obtain (2.2) and their conjugates and also

$$\left.\begin{aligned} \frac{1}{4\pi}\Box \mathrm{V} &= e\sum_1^4 \psi_\lambda^*\psi_\lambda \\ \frac{1}{4\pi}\Box \mathrm{A}_1 &= -e\{\psi_1^*\psi_4+\psi_2^*\psi_3+\psi_3^*\psi_2+\psi_4^*\psi_1\} \end{aligned}\right\}, \tag{3.2}$$

etc., where $\Box$ signifies $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$. We conclude that the electromagnetic effect of the electron can be represented by taking density ρ and current densities j_1, j_2, j_3, where

$$\left.\begin{aligned} \rho &= -e(\psi_1^*\psi_1+\psi_2^*\psi_2+\psi_3^*\psi_3+\psi_4^*\psi_4) \\ j_1 &= ce(\psi_1^*\psi_4+\psi_2^*\psi_3+\psi_3^*\psi_2+\psi_4^*\psi_1) \\ j_2 &= ce(-i\psi_1^*\psi_4+i\psi_2^*\psi_3-i\psi_3^*\psi_2+i\psi_4^*\psi_1) \\ j_3 &= ce(\psi_1^*\psi_3-\psi_2^*\psi_4+\psi_3^*\psi_1-\psi_4^*\psi_2) \end{aligned}\right\}, \tag{3.3}$$

provided that the ψ's are normalised by the rule

$$\iiint \sum_{\lambda=1}^4 \psi_\lambda\psi_\lambda^*\,dx\,dy\,dz = 1. \tag{3.4}$$

Since S is invariant for relativity transformations, ρ and j will be covariant for such transformations, and this can also be easily verified by applying the transformations of § 2 in turn.

It should be observed that ρ and j satisfy the equation of continuity

$$\frac{\partial\rho}{\partial t}+\operatorname{div} j = 0, \tag{3.5}$$

as may be directly verified with the use of (2.2). That (3.5) should be verified is in a sense the starting point of Dirac's argument. For if it had not been so spontaneously, we should have been compelled to force it by introducing into S a term $\left(\frac{1}{c}\frac{\partial \mathrm{V}}{\partial t}+\operatorname{grad}\mathrm{A}\right)\mathrm{F}$ with F undetermined. The result would give extra terms in ρ and j involving $\frac{\partial \mathrm{F}}{\partial t}$ and grad F, and the condition $\frac{1}{c}\frac{\partial \mathrm{V}}{\partial t}+\operatorname{grad}\mathrm{A}=0$ would then fix F. In general it would involve V and A, and therefore ρ and j would do so too. It was the presence of such terms in Klein's†

† 'Z. f. Physik,' vol. 41, p. 407 (1927).

evaluation of density and current that was objectionable and that led Dirac to his new equations.

To complete the rules for calculating intensities we have to break up ρ and j into terms corresponding to pairs of states. This can be done in the manner of Klein, but perhaps the following picture, though very incomplete,† may make the process clearer, and may show under what conditions we expect to get line spectra with definite intensities ; it is applicable to any system. Imagine that we have an assembly of atoms in an enclosure. The equations (2.2) and (3.2), together with appropriate boundary conditions, will describe the state of affairs. Thermal equilibrium will be produced, with the accompanying black radiation, and the equations will be quite insoluble, because in solving (2.2) the electromagnetic fields, themselves determined by (2.2), will not be small. At any instant of time we can imagine the state expressed by an expansion in proper functions, and the average values of the coefficients will be determined by the appropriate statistics—in particular, states of nearly equal energy will have equal average coefficients. Now if the enclosing barrier is suddenly removed, the radiation before present will spread away with the speed of light and the matter will be left only under the influence of any existing permanent electromagnetic forces. The problem is now soluble by approximations, first solving (2.2) for the ψ's, neglecting the radiation, and then substituting the values found in (3.2) to give the radiation. If ψ_λ is initially $\sum_p a_p \psi_\lambda^p$, the first process gives $\psi_\lambda = \sum_p a_p \psi_\lambda^p e^{-i\frac{2\pi}{h} W_p t}$. Next we form

$$\rho = \sum_p \sum_q \sum_{\lambda=1}^{4} a_p a_q^* \psi_\lambda^p \psi_\lambda^{q*} e^{i\frac{2\pi}{\lambda}(W_q - W_p)t},$$

and similar values for the j's. Substituting these in (3.2) we can evaluate the electric force at a distant point, and it will evidently consist of a sum of periodic terms corresponding to the spectrum lines given by $W_p \rightarrow W_q$. The process is exactly that introduced by Klein, only a little more definite in that no appeal is made to the correspondence principle.

4. We shall next exhibit the relationship of Dirac's equations to previous theories, and shall show that the latter are successive approximations to (2.2). The guiding principle in this is the fact that of the four ψ's, ψ_3 and ψ_4 are very much larger than ψ_1, ψ_2, since this leads to a method of approximation. We shall treat of the case of the stationary states of an electron in an atom.

The first approximation leads to Schrödinger's equation in both ψ_3 and ψ_4

† The incompleteness lies chiefly in the fact that no distinction is made between one atom and many atoms.

independently; but in doing this it must not be forgotten that that equation is not even approximately right as far as concerns the effects of magnetic fields. We must therefore restrict ourselves to equations in which $p_1 = \frac{h}{2\pi i}\frac{\partial}{\partial x}$ without a vector potential, so that $p_1 p_2 = p_2 p_1$, etc. Starting with complete neglect of ψ_1, ψ_2 we see that ψ_3, ψ_4, and therefore ψ_1, ψ_2 also, are proportional to $e^{-i\frac{2\pi}{h}mc^2 t}$. Hence $(p_0 + mc)\,\psi_1$ is nearly equal to $2mc\psi_1$, and so we take

$$\left.\begin{aligned}\psi_1 &= -\frac{h}{2\pi i}\frac{1}{2mc}\left[\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\psi_4 + \frac{\partial}{\partial z}\psi_3\right]\\ \psi_2 &= -\frac{h}{2\pi i}\frac{1}{2mc}\left[\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi_3 - \frac{\partial}{\partial z}\psi_4\right]\end{aligned}\right\}. \tag{4.1}$$

Substituting in the third equation we get

$$\left(-mc + \frac{eV}{c} - \frac{h}{2\pi i}\frac{1}{c}\frac{\partial}{\partial t}\right)\psi_3 = \left(\frac{h}{2\pi i}\right)^2 \frac{1}{2mc}\Delta\psi_3 .$$

The same equation holds for ψ_4, so that we simply have Schrödinger's equation twice over.

In the second approximation, following Dirac's § 4, we form exact second order equations in ψ_3, ψ_4, and from these we eliminate $\psi_1\,\psi_2$ by means of (4.1). In (2.2) operate the third equation by $(mc + p_0)$, the second by $-(p_1 - ip_2)$, the first by $-p_3$ and add. We have

$$p_0 p_1 - p_1 p_0 = -\frac{h}{2\pi i}\frac{1}{c}\frac{\partial}{\partial t}\frac{eA_1}{c} - \frac{h}{2\pi i}\frac{\partial}{\partial x}\frac{eV}{c} = \frac{e}{c}\frac{h}{2\pi i}E_1,$$

$$p_1 p_2 - p_2 p_1 = \frac{h}{2\pi i}\frac{\partial}{\partial x}\frac{eA_2}{c} - \frac{h}{2\pi i}\frac{\partial}{\partial y}\frac{eA_1}{c} = \frac{e}{c}\frac{h}{2\pi i}H_3,$$

and using these and similar relations we get

$$\left[-m^2c^2 + {p_0}^2 - {p_1}^2 - {p_2}^2 - {p_3}^2 - \frac{e}{c}\frac{h}{2\pi}H_3\right]\psi_3 - \frac{e}{c}\frac{h}{2\pi}(H_1 - iH_2)\,\psi_4$$
$$- \frac{e}{c}\frac{h}{2\pi}i\left[(E_1 - iE_2)\,\psi_2 + E_3\psi_1\right] = 0.$$

Now substitute for $\psi_1\,\psi_2$ from (4.1). Those equations ought now to contain terms involving the A's, but, as they are to be multiplied by E's, these may be neglected. We have

$$\left(D - \frac{2\pi e}{ch}H_3\right)\psi_3 - \frac{2\pi e}{ch}(H_1 - iH_2)\,\psi_4 + \frac{e}{2mc^2}\left\{\left(E_1\frac{\partial}{\partial x} + E_2\frac{\partial}{\partial y} + E_3\frac{\partial}{\partial z}\right)\psi_3\right.$$
$$\left. + i\left(E_2\frac{\partial}{\partial z} - E_3\frac{\partial}{\partial y}\right)\psi_4 + \left(E_3\frac{\partial}{\partial x} - E_1\frac{\partial}{\partial z}\right)\psi_4 + i\left(E_1\frac{\partial}{\partial y} - E_2\frac{\partial}{\partial x}\right)\psi_3\right\} = 0, \tag{4.2}$$

where $D = \left(\frac{2\pi}{h}\right)^2(-m^2c^2 + p_0^2 - p_1^2 - p_2^2 - p_3^2)$ is the same quantity as in my paper. Similarly

$$\left(D + \frac{2\pi e}{ch} H_3\right)\psi_4 - \frac{2\pi e}{ch}(H_1 + iH_2)\psi_3 + \frac{e}{2mc^2}\left\{\left(E_1\frac{\partial}{\partial x} + E_2\frac{\partial}{\partial y} + E_3\frac{\partial}{\partial z}\right)\psi_4 + i\left(E_2\frac{\partial}{\partial z} - E_3\frac{\partial}{\partial y}\right)\psi_3 - \left(E_3\frac{\partial}{\partial x} - E_1\frac{\partial}{\partial z}\right)\psi_3 - i\left(E_1\frac{\partial}{\partial y} - E_2\frac{\partial}{\partial x}\right)\psi_4\right\} = 0. \tag{4.3}$$

Apart from the terms in

$$P \equiv E_1\frac{\partial}{\partial x} + E_2\frac{\partial}{\partial y} + E_3\frac{\partial}{\partial z}, \tag{4.4}$$

these are identical with the equations of my paper, provided that we identify ψ_3 with f, ψ_4 with $-g$. The extra terms in P rectify one of the earlier defects, for with my equations the s-levels of hydrogen fell in the wrong place though all others were correct. Now when the approximative method is used, the new terms affect the levels by an amount depending on $\int_0^\infty f(r)\,[P\,.\,f(r)]\,r^2\,dr$ and for hydrogen $E_1 = ex/r^3$ so that $P = \frac{e}{r^2}\frac{d}{dr}$ and the integral depends on $[f(0)]^2$. Since $f(0)$ vanishes unless $k = 0$, all levels other than s-levels are unaffected, and a more detailed calculation shows that the s-levels now fall in the right place.

The formulæ for intensities are also the same to a first approximation, because to this degree $ff^* + gg^*$ is the same as $\sum_1^4 \psi_\lambda\psi_\lambda^*$. It is of more interest to consider the formulæ for magnetic moment. For this we take all the ψ's in (3.3) as belonging to a single state, so that the time disappears and we have $div\ j = 0$. With this condition and the condition $\iiint j\,dx\,dy\,dz = 0$ (which holds because there is no progressive current in a stationary atom) the magnetic moment can be seen to have as first component

$$\mu_1 = \frac{1}{2c}\iiint (yj_3 - zj_2)\,dx\,dy\,dz.$$

Substitute (4.1) in (3.3) so as to obtain the approximations for j_2, j_3. After some partial integrations we find

$$\mu_1 = \frac{eh}{4\pi mc}\iiint\left[\frac{1}{2i}\{\psi_3 R_1\psi_3^* + \psi_4 R_1\psi_4^* - \psi_3^* R_1\psi_3 - \psi_4^* R_1\psi_4\} - (\psi_3\psi_4^* + \psi_3^*\psi_4)\right]dx\,dy\,dz, \tag{4.5}$$

where $R_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$. This is the value† given before, when we replace ψ_3, ψ_4 by f and $-g$. Similar methods apply for the other components. A cyclic change gives the first terms, and the last are

$$-i(\psi_3\psi_4^* - \psi_3^*\psi_4) \quad \text{and} \quad +(\psi_4\psi_4^* - \psi_3\psi_3^*) \tag{4.6}$$

respectively for μ_2 and μ_3.

Dirac's success in finding the accurate equations shows the great superiority of principle over the previous empirical method, but it is perhaps not without interest (at any rate to the present writer, who had projected but not begun such work) to consider whether the empirical method could have led by way of improved approximations to the accurate result. The most critical step in doing so had been made, though not quite rightly and for a wrong reason, in the replacement of $2mc$ by a time differential (it had been replaced by

$$2\left(-\frac{h}{2\pi i}\frac{1}{c}\frac{\partial}{\partial t} + \frac{eV}{c}\right),$$

whereas only half should have been treated in this way). There also seemed nothing to prevent carrying out a higher approximation so as to make the hydrogen levels fall more exactly together. A further guide lay in the fact that the electric current *must* be a more primitive thing than the magnetic moment, and when the current is deduced from (4.5), (4.6), it has certain complicated small terms admitting of modification. On the other hand, the absence of the terms (4.4) would have caused trouble. On the whole, it seems not impossible that one might with much labour have arrived at some sort of eliminant of Dirac's equations. Fortunately, he has made such work unnecessary.

5. The free motion of an electron calls for comment. In the equations (2.2) we now omit V and A altogether. Assume as solution

$$\psi_\lambda = a_\lambda \exp i\frac{2\pi}{h}(px + qy + rz - Wt),$$

and on substituting we have a determinant which reduces to

$$(W^2 - m^2c^2 - p^2 - q^2 - r^2)^2 = 0.$$

† In § 6 of my paper, by a blunder I took the magnetic moment as $\frac{W}{\partial H}$ instead of $-\frac{\partial W}{\partial H}$. The expressions (6.7) there for the components of μ should have their signs changed. Those expressions can be written in terms of f, g by substituting $X_1 = f$, $X_2 = if$, $X_3 = g$, $X_4 = ig$.

We must therefore take $W = \sqrt{\{m^2c^2 + p^2 + q^2 + r^2\}}$, which we shall call W_{pqr} for short. We can take a_3, a_4 as arbitrary, say, A, B, and then have

$$a_1 = -\frac{Ar + B(p - iq)}{mc + W_{pqr}}, \qquad a_2 = -\frac{A(p + iq) - Br}{mc + W_{pqr}}.$$

From these we find that ρ and j are proportional to $|A|^2 + |B|^2$, and that j is a vector along $p : q : r$.

An important point now arises if we consider the problem from the point of view raised in a recent paper.† The motion of the electron is there regarded as a pure wave problem, the solution consisting in finding the way in which given *arbitrary* initial conditions are propagated. Suppose that we have arbitrary initial values of all four ψ's at every point of space. We can submit them to Fourier analysis and have

$$\psi_\lambda = \int a_\lambda(p, q, r) \exp i\frac{2\pi}{h}(px + qy + rz)\, dp\, dq\, dr.$$

The a_λ's will have arbitrary values, but this is impossible since we have just seen that a_1, a_2 are determinate in terms of a_3, a_4. There can hardly be a question that a complete theory will overcome this difficulty by admitting negative values of W, but we are evidently in contact with the question raised by Dirac in his § 1, connected with the possible changes of e into $-e$. At present this is unsolved, so we must be content to say that we are not entitled to assume completely arbitrary initial conditions, but may only take two of the four functions arbitrary.

To understand the physical meaning of the equations in free space, we want to be able to associate a given solution with the rectilinear motion of an electron with magnetic moment in a given direction. As long as we only deal with solutions of type (5.1) nothing can be said about the magnetic moment, because the waves fill all space uniformly and there is, therefore, no distant point left from which to observe it. In order to get a magnetic moment we must construct a wave packet. We may, for instance, assume that initially ψ_3 and ψ_4 contain a factor $\exp -\frac{1}{2\sigma^2}(x^2 + y^2 + z^2)$. These will fix ψ_1, ψ_2, and it is important to observe that in consequence of the differential inter-relation between them, the ratio of $\psi_1 : \psi_3$ will vary in the different parts of the packet, so that the current j will no longer be everywhere straight along the main direction of wave motion, and a magnetic moment becomes possible. The complete solution can be set down in Fourier integrals, but it does not seem possible to

† 'Roy. Soc. Proc.,' A, vol. 117, p. 258 (1927).

work them out. It is therefore simpler to recur to the approximation of § 4, which shows that coefficients A and B for ψ_3, ψ_4 will be associated with various moments. By consideration of the last term in (4.5), (4.6), we see that if A, B are respectively proportional to $(1 - n)$ and $(-l - im)$ we shall have magnetic moment along the direction $l : m : n$. For high speeds of motion the approximation would fail, but so would the idea of magnetic moment.

The same approximate method is adequate for the case of motion in a magnetic field and for the Stem Gerlach effect. For the case of a uniform field it is possible to find accurate solutions of the equations, but they correspond to a quantised circular motion, and are not of much interest, as they need to be combined into wave packets if any close relationship is to be seen with the velocity of the electron.

6. We now consider the energy levels for an electron in a central field, and in particular for hydrogen. Before proceeding to the solution, it will be well to discuss the question of the nomenclature of the various quantum states. Dirac points out that angular momentum is no longer an integral of the motion, but finds a modified integral of a similar type. He thus suppresses the use of k and uses a quantum number j. With the method of solution which we shall be using the dynamical meaning of quantum numbers goes very much into the background, and we are left only with integers defining the orders of spherical harmonics and other functions entering the solution. From this point of view the quantum number is only a convenient name associated with those functions —for example the quantum number m was adopted as $u + \frac{1}{2}$ in the earlier theory because a certain solution involved P_k^u and P_k^{u+1}. The fact that there exists a dynamical integral is then largely irrelevant—under special circumstances it might help in guessing a solution, but usually it merely reduces to an identity in the properties of a function found in some other way.

In view of these considerations, I have concluded, with some hesitation, that it is more convenient not to alter the notation in the way done by Dirac. The quantum numbers k, j, m retain their old classificatory, but not their dynamical significance; in this way any line of a spectrum can be described by the same numbers as were used before and doublets are classified like other multiplets. To define them more precisely we take m as given by Landé's g-formula, and j as the maximum positive value of m. For doublets it is therefore a half-number, and k may be either of the adjacent integers. We shall, in fact, see that every solution involves four different spherical harmonics, P_k^u, P_k^{u+1}, P_{k+1}^u, P_{k+1}^{u+1} and j and m are simply the averages of subscripts and of superscripts respectively. k, which is the subscript of ψ_3, has not the same symmetrical property as j

and m, but is called a quantum number because it defines the rule of combination of levels and of the Zeeman effect. We adopt for it the numeration 0, 1, 2, ..., not 1, 2, 3, ..., as this has proved most convenient in spectroscopy.

In Dirac's new notation j can take on negative values, and it is at first sight tempting to follow this change. It would not be hard to invent negative spherical harmonics P^u_{-j}, which would make it possible to write all the solutions in a single form, but it would lead to little simplification. For his j values do not run right through from positive to negative; they miss the value zero and it would be necessary to study combinations of like and of unlike signs separately, and this can be done just as well with only positive values and a second number k acting as a sort of plus or minus sign. Moreover, as we shall see, we can take full advantage of Dirac's method in discussing the radial functions. The following scheme shows the relationship between the values used here and by Dirac:—

		s.	*p.*		*d.*	
Here	k	0	1	1	2	2
	j	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
Dirac	j	-1	1	-2	2	-3

7. In order to solve for the levels in a radial field of force, we put $p_0 = \frac{W}{c} + \frac{eV}{c}$ where V depends only on the radius, and we omit the vector potentials. The equations (2.2) now become:—

$$\left.\begin{aligned}
\frac{2\pi i}{h}\left(\frac{W+eV}{c}+mc\right)\psi_1+\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\psi_4+\frac{\partial}{\partial z}\psi_3=0\\
\frac{2\pi i}{h}\left(\frac{W+eV}{c}+mc\right)\psi_2+\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_3-\frac{\partial}{\partial z}\psi_4=0\\
\frac{2\pi i}{h}\left(\frac{W+eV}{c}-mc\right)\psi_3+\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\psi_2+\frac{\partial}{\partial z}\psi_1=0\\
\frac{2\pi i}{h}\left(\frac{W+eV}{c}-mc\right)\psi_4+\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_1-\frac{\partial}{\partial z}\psi_2=0
\end{aligned}\right\}. \qquad (7.1)$$

Following previous methods we try to express the four functions as spherical harmonics multiplied by radial functions.

We may first conveniently give certain formulæ for spherical harmonics

that are easily proved. We abbreviate the notation by writing P_k^u for the whole harmonic, thus

$$\mathrm{P}_k^u = (k-u)!\sin^u\theta\left(\frac{d}{d\,.\cos\theta}\right)^{k+u}\frac{(\cos^2\theta-1)^k}{2^k\,.\,k!}e^{iu\phi},$$

which exists for any positive integral value of k and for any integral value of u between $\pm k$ inclusive. Then, if f is any function of the radius we have:—

$$\left.\begin{aligned}
\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)f\mathrm{P}_k^u &= \frac{1}{2k+1}\left\{\left(\frac{df}{dr}-\frac{k}{r}f\right)\mathrm{P}_{k+1}^{u+1}\right.\\
&\qquad\left.-(k-u)(k-u-1)\left(\frac{df}{dr}+\frac{k+1}{r}f\right)\mathrm{P}_{k-1}^{u+1}\right\}\\
\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)f\mathrm{P}_k^u &= \frac{1}{2k+1}\left\{-\left(\frac{df}{dr}-\frac{k}{r}f\right)\mathrm{P}_{k+1}^{u-1}\right.\\
&\qquad\left.+(k+u)(k+u-1)\left(\frac{df}{dr}+\frac{k+1}{r}f\right)\mathrm{P}_{k-1}^{u-1}\right\}\\
\frac{\partial}{\partial z}f\mathrm{P}_k^u &= \frac{1}{2k+1}\left\{\left(\frac{df}{dr}-\frac{k}{r}f\right)\mathrm{P}_{k+1}^{u}\right.\\
&\qquad\left.+(k+u)(k-u)\left(\frac{df}{dr}+\frac{k+1}{r}f\right)\mathrm{P}_{k-1}^{u}\right\}
\end{aligned}\right\}. \quad (7.2)$$

We may note that these formulæ automatically look after "end effects"; thus, if we apply the first or third to P_k^k the factor $(k-u)$ in the second term cuts out the terms in P_{k-1}^{k+1}, P_k^{k+1}, functions which do not exist.

Consider how these relations will work in (7.1). We try to get a solution in which at least one of the ψ's only involves a single spherical harmonic. Suppose that ψ_3 is proportional to P_k^u. Then the third equation of (7.1) tells us that terms from $\frac{\partial}{\partial z}\psi_1$ and $\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\psi_2$ must cancel out with this, and any other terms they give must cancel out together. It follows that ψ_1, ψ_2 must involve the same function of r, and must either be of the forms P_{k+1}^u, P_{k+1}^{u+1}, or P_{k-1}^u, P_{k-1}^{u+1}. A similar argument then shows that ψ_4 must involve r in the same way as ψ_3 and must have spherical function P_k^{u+1}. We therefore take as trial solution

$$\psi_1 = -ia_1\,\mathrm{F}(r)\,\mathrm{P}_{k+1}^u, \qquad \psi_2 = -ia_2\,\mathrm{F}(r)\,\mathrm{P}_{k+1}^{u+1},$$

$$\psi_3 = a_3\,\mathrm{G}(r)\,\mathrm{P}_k^u, \qquad \psi_4 = a_4\,\mathrm{G}(r)\,\mathrm{P}_k^{u+1}$$

(the factor $-i$ is introduced in ψ_1, ψ_2 to make F real). We then find that the

a's can be adjusted so that all four equations are satisfied. For example, the first equation gives

$$\frac{2\pi}{h}\left(\frac{W+eV}{c}+mc\right)a_1 F P^u_{k+1} + \frac{a_4}{2k+1}\left\{-\left(\frac{dG}{dr}-\frac{k}{r}G\right)P^u_{k+1}+(k+u+1)(k+u)\left(\frac{dG}{dr}+\frac{k+1}{r}G\right)P^u_{k-1}\right\} + \frac{a_3}{2k+1}\left\{\left(\frac{dG}{dr}-\frac{k}{r}G\right)P^u_{k+1}+(k+u)(k-u)\left(\frac{dG}{dr}+\frac{k+1}{r}G\right)P^u_{k-1}\right\}=0. \quad (7.3)$$

If, then, we take $a_4(k+u+1)+a_3(k-u)=0$, the terms in P^u_{k-1} cancel. In the second equation, the same ratio makes the coefficient of P^{u+1}_{k-1} disappear. From the other two equations we find similarly $a_1=a_2$; the ratio $a_1:a_3$ is immaterial as it may be incorporated in F : G. We thus find as the complete solution

$$\left.\begin{aligned} \psi_1 &= -iF_kP^u_{k+1}, & \psi_2 &= -iF_kP^{u+1}_{k+1} \\ \psi_3 &= (k+u+1)\,G_kP^u_k, & \psi_4 &= (-k+u)\,G_kP^{u+1}_k \end{aligned}\right\}, \quad (7.4)$$

where F_k, G_k satisfy the relations

$$\left.\begin{aligned} \frac{2\pi}{h}\left(\frac{W+eV}{c}+mc\right)F+\frac{dG}{dr}-\frac{k}{r}G=0 \\ -\frac{2\pi}{h}\left(\frac{W+eV}{c}-mc\right)G+\frac{dF}{dr}+\frac{k+2}{r}F=0 \end{aligned}\right\}. \quad (7.5)$$

This solution we name $(k, j=k+\frac{1}{2}, m=u+\frac{1}{2})$.

A similar process gives a different solution if we make the first instead of the second terms in (7.3) cut out. We now have ψ_1 involving P^u_{k-1} instead of P^u_{k+1}. F and G must then satisfy

$$\left.\begin{aligned} \frac{2\pi}{h}\left(\frac{W+eV}{c}+mc\right)F+\frac{dG}{dr}+\frac{k+1}{r}G=0 \\ -\frac{2\pi}{h}\left(\frac{W+eV}{c}-mc\right)G+\frac{dF}{dr}-\frac{k-1}{r}F=0 \end{aligned}\right\}. \quad (7.6)$$

We can regard these equations as the same as (7.5) by changing k into $k-1$; so we write the solution F_{-k-1}, G_{-k-1}. Then we have

$$\left.\begin{aligned} \psi_1 &= -i(k+u)\,F_{-k-1}P^u_{k-1}, & \psi_2 &= -i(-k+u+1)\,F_{-k-1}P^{u+1}_{k-1} \\ \psi_3 &= G_{-k-1}P^u_k & \psi_4 &= G_{-k-1}P^{u+1}_k \end{aligned}\right\} \quad (7.7)$$

This solution we name $(k, j=k-\frac{1}{2}, m=u+\frac{1}{2})$. We see in the subscripts of F and G the point of Dirac's method of allowing j to be negative. The equations (7.5) and (7.6) are substantially his equations at the foot of p. 622.

We have not justified the use of k as a quantum number, and this cannot be done until we study intercombinations; but anticipating this we may now count up the number of solutions and see that it is right for doublet spectra. In order to do this we must see what end cases are admissible. Take the first type of solution (7.4). If $u = k$, $\psi_4 = 0$ on account of the factor $(-k+u)$, and so in substituting in (7.1) we shall not be led astray by applying (7.2) to the impermissible function P_k^{k+1}. On the other hand, if we take $u = k+1$, we get a function ψ_2, but none of the others, and so evidently no solution. Similarly, at the other end we may take $u = -k-1$ (involving $\psi_3 = 0$) but not $u = -k-2$. In all there will be $2k+2$ solutions. In the second type (7.7) we see in the same way that we may take u between $-k$ and $k-1$ inclusive and so get $2k$ solutions. In both cases there are therefore $2j+1$ solutions, as there should be. In the special case $k = 0$ there are only two solutions of the first type, and none of the second.

We have thus found by trial a system of solutions of our equations, and the important question arises as to whether it is a *complete* set. Can we simultaneously expand four arbitrary functions $\psi_1 \ldots \psi_4$ in terms of the solutions (7.4) and (7.7)? The full consideration would require a discussion of the radial functions including the quasi-hyperbolic case, which we shall not attempt; for even without it we can see that we have only half as many proper functions as are required. Taking an arbitrary radius we may expand the four given functions in spherical harmonics over the sphere. The kth harmonic will thus have $4(2k+1)$ coefficients, all arbitrary; whereas we have seen that there are only $2k+2+2k$ proper functions with ψ_3 and ψ_4 of order† k. The incompleteness is evidently the same thing as was pointed out in § 5. To get a complete set we must double the number of solutions by admitting negative values of the energy, and we have at present little idea of what this means.

8. We now discuss the radial functions (7.5). In the case where the radial force is arbitrary we can proceed by approximation based on the fact that F is much smaller than G. But the process would run very closely parallel to that of Dirac (p. 623) and we need not give it. We may only note that the null approximation gives Schrödinger's equation for G, and the next breaks it into

† If we start at the zero order and work up, determining each term as we go, the counting is a little different. For example, say that the expansion of the ψ's only contains zero and first order harmonics. Then we have $2+2=4$ relations to fit to $4(1+3)=16$ arbitrary quantities. If the second order is included as well, we have $2+2+4+4=12$ relations for $4(1+3+5)=36$ quantities. In an *infinite* series, of course, the exact counting does not matter; the two sequences approximate to the ratio 1 : 2.

two terms depending on parameters k and $-k-1$, in fact, the ordinary doublet spectrum.

We shall therefore proceed to find the accurate solution for the case of hydrogen. We take $V = e/r$ and it is convenient to introduce certain auxiliary symbols. Take

$$\frac{2\pi}{h}\left(mc + \frac{W}{c}\right) = A^2, \qquad \frac{2\pi}{h}\left(mc - \frac{W}{c}\right) = B^2, \tag{8.1}$$

with A and B both positive, and write as usual for the "fine structure constant"

$$\gamma = \frac{2\pi e^2}{ch} \tag{8.2}$$

and the equations (7.5) become

$$\left.\begin{aligned} \left(A^2 + \frac{\gamma}{r}\right) F + \frac{dG}{dr} - \frac{k}{r} G = 0 \\ \left(B^2 - \frac{\gamma}{r}\right) G + \frac{dF}{dr} + \frac{k+2}{r} F = 0 \end{aligned}\right\}. \tag{8.3}$$

We solve these in series of the form

$$\left.\begin{aligned} F = e^{-\lambda r}\{a_0 r^\beta + a_1 r^{\beta-1} + a_2 r^{\beta-2} + \ldots\} \\ G = e^{-\lambda r}\{b_0 r^\beta + b_1 r^{\beta-1} + b_2 r^{\beta-2} + \ldots\} \end{aligned}\right\}. \tag{8.4}$$

Substitute and equate to zero the various terms. We have

$$\begin{aligned} A^2 a_0 - \lambda b_0 = 0, \qquad & B^2 b_0 - \lambda a_0 = 0, \\ A^2 a_1 + \gamma a_0 - \lambda b_1 + (\beta - k) b_0 = 0, \qquad & B^2 b_1 - \gamma b_0 - \lambda a_1 + (\beta + k + 2) a_0 = 0, \\ \cdot \quad \cdot \quad \cdot \qquad & \cdot \quad \cdot \quad \cdot \\ A^2 a_{s+1} + \gamma a_s - \lambda b_{s+1} + (\beta - k - s) b_s = 0, \qquad & B^2 b_{s+1} - \gamma b_s - \lambda a_{s+1} + (\beta + k + 2 - s) a_s = 0. \end{aligned}$$

The first pair determine $\lambda = AB$; we must take the positive solution to make F, G finite at infinity. We also have $b_0/a_0 = A/B$. Substituting in the next pair, we find that both a_1 b_1 can be eliminated simultaneously. We get the indicial equation

$$\beta = -1 + \gamma \frac{A^2 - B^2}{2AB}. \tag{8.5}$$

A similar elimination of a_{s+1}, b_{s+1} can be carried out and we have

$$A a_s\left(\beta + k + 2 - s + \gamma \frac{B}{A}\right) + B b_s\left(\beta - k - s - \gamma \frac{A}{B}\right) = 0,$$

from which we may substitute

$$\left.\begin{aligned} b_s &= c_s\left[\gamma\frac{B}{A}+\beta+k+2-s\right] \\ a_s &= c_s\,.\,\frac{B}{A}\left[\gamma\frac{A}{B}-\beta+k+s\right] \end{aligned}\right\} \tag{8.6}$$

and can now form a difference equation for c_s. This reduces to

$$AB\,(2s+2)\,c_{s+1} = -\,c_s\{(s-\beta-1)^2-[(k+1)^2-\gamma^2]\}.$$

Writing $\sqrt{(k+1)^2-\gamma^2} = k'$ (supposed positive) we have

$$2AB\,(s+1)\,c_{s+1} = -\,c_s\,(\beta+1-s-k')\,(\beta+1-s+k'),$$

and so

$$c_s = (-)^s\frac{(\beta-k'+1)(\beta-k')\ldots(\beta-k'-s+2)\,(\beta+k'+1)(\beta+k')\ldots(\beta+k'-s+2)}{2^s\,.\,s\,!\,(AB)^s}. \tag{8.7}$$

The series for F and G are composed of terms of this type each multiplied by a factor given by (8.6). If the solution is to be finite throughout space it is necessary that these series should terminate for some value of s such that $\beta - s \geqslant 0$. It is therefore necessary that $\beta = k' + n' - 1$, where n' is zero or a positive integer. This condition determines the energy levels. For we have

$$k'+n' = \gamma\frac{A^2-B^2}{2AB} = \gamma\frac{W}{\sqrt{\{m^2c^4-W^2\}}},$$

and so

$$W = mc^2\left\{1+\frac{\gamma^2}{(k'+n')^2}\right\}^{-\frac{1}{2}}. \tag{8.8}$$

This is exactly the original Sommerfeld expression for the energy levels of hydrogen. The only difference is that our k may take the value zero, so that the formula now involves $\sqrt{\{(k+1)^2-\gamma^2\}}$.

The process of solution has at no stage made use of the fact that k is positive, and we conclude that the same solution will hold for (7.6) provided that we write $-k-1$ for k. In this case then $k' = \sqrt{(k^2-\gamma^2)}$, and we see how the two k levels split. If we compare j's instead of k's we must take $k+1$ for the second solution, and the levels fall exactly together. In the case $k = 0$ there is no second solution and no corresponding radial function for $k = -1$, as (8.7) will not then factorise into real factorials, so that the series cannot terminate.

We may now express the proper functions in terms of the quantum numbers. We make use of the auxiliary quantities

$$k' = \sqrt{\{(k+1)^2 - \gamma^2\}}, \qquad \mathrm{N} = \sqrt{\{(k'+n')^2 + \gamma^2\}}$$

and

$$\frac{2\pi mc\gamma}{h} = \frac{4\pi^2 mc^2}{h^2} = \frac{1}{a}.$$

Then a is the "radius of the first hydrogen orbit" and N is approximately the total quantum number, counting 1, 2, 3, ... for the hydrogen levels. We then have

$\mathrm{W} = mc^2(k'+n')/\mathrm{N}$

$$\mathrm{G}_k = e^{-r/a\mathrm{N}}\Big\{r^{k'+n'-1}(\mathrm{N}+k+1) - r^{k'+n'-2}a\mathrm{N}(\mathrm{N}+k)\frac{n'(n'+2k')}{2}$$
$$+ r^{k'+n'-3}a^2\mathrm{N}^2(\mathrm{N}+k-1)\frac{n'(n'-1)(n'+2k')(n'+2k'-1)}{2\,.\,4} - \ldots\Big\}$$

$$\mathrm{F}_k = \frac{\gamma}{\mathrm{N}+k'+n'}e^{-r/a\mathrm{N}}\Big\{r^{k'+n'-1}(\mathrm{N}+k+1) - r^{k'+n'-2}a\mathrm{N}(\mathrm{N}+k+2)\frac{n'(n'+2k')}{2}$$
$$+ r^{k'+n'-3}a^2\mathrm{N}^2(\mathrm{N}+k+3)\frac{n'(n'-1)(n'+2k')(n'+2k'-1)}{2\,.\,4} - \ldots\Big\}. \qquad (8.9)$$

We may observe that if we approximate by neglecting γ, we find that G_k and G_{-k-1} are respectively $(\mathrm{N}+k+1)$ and $(\mathrm{N}-k)$ times the ordinary radial function of Schrödinger.

The solution reveals a small blemish in the equations, for we have to admit the existence of proper functions which become infinite. The last term in the series for G has power $r^{k'-1}$, and if $k=0$, k' is very slightly less than 1, so that there will be a term with a small negative power of r. Of course all integrals connected with the spectrum are amply convergent. We do not perhaps know enough about the essential rules for proper functions to pay much attention to this defect. Moreover, it may well be that it would disappear if we could solve the problem of two bodies properly instead of treating the nucleus as an abstract centre of force.

As an example of these apparently complicated functions, we may set down the solutions corresponding to $\mathrm{N}=1$ (exactly), the lowest hydrogen state.

We shall replace the spherical harmonic symbols by the corresponding solid harmonics. Then the two solutions are ($k = 0$, $n' = 0$, $\mathrm{N} = 1$, $u = 0$ and -1)

$$\mathrm{W} = mc^2\sqrt{(1-\gamma^2)}, \qquad \beta = \sqrt{(1-\gamma^2)} - 1,$$

$$\psi_1 = -\frac{i\gamma}{1+\sqrt{(1-\gamma^2)}} z r^{\beta-1} e^{-r/a},$$

$$\psi_2 = -\frac{i\gamma}{1+\sqrt{(1-\gamma^2)}} (x+iy) r^{\beta-1} e^{-r/a},$$

$$\psi_3 = r^\beta e^{-r/a}, \qquad \psi_4 = 0,$$

and

$$\psi_1 = \frac{i\gamma}{1+\sqrt{(1-\gamma^2)}} (x-iy) r^{\beta-1} e^{-r/a},$$

$$\psi_2 = -\frac{i\gamma}{1+\sqrt{(1-\gamma^2)}} z r^{\beta-1} e^{-r/a},$$

$$\psi_3 = 0, \qquad \psi_4 = -r^\beta e^{-r/a}.$$

This will suffice as an illustration of the accurate solution.

9. We now consider the rules of combination. The emission can be calculated from (3.2) by setting down the values of V and A at a distant point. They depend on the retarded potentials and the work follows that of Klein very closely, so that we need not give details. We omit discussion of the very weak radiations of quadrupole and higher types. In calculating the chief radiation we shall have contributions from V and also from A. Taking the transition $p \to q$ we write $\mathrm{W}_p - \mathrm{W}_q = h\nu$ and find

$$\mathrm{V}(x', y', z') = \frac{e^{i2\pi\nu(t-r'/c)}}{r'} \iiint \rho_{pq} \frac{(\mathbf{r}, \mathbf{r}')}{r'} i\frac{2\pi\nu}{c}\, dx\, dy\, dz,$$

where $\mathbf{r}$, $\mathbf{r}'$ are the vectors from the origin to x, y, z and to the distant point of observation x', y', z' respectively, and r' is the absolute value of $\mathbf{r}'$. Similarly,

$$\mathrm{A}_1(x', y', z') = \frac{e^{i2\pi\nu(t-r'/c)}}{r'} \iiint \frac{(j_1)_{pq}}{c}\, dx\, dy\, dz.$$

V involves the electric moment, and it should be noticed that, in spite of its different appearance, A does so too. For

$$\iiint j_1\, dx\, dy\, dz = \iiint -x\left(\frac{\partial j_1}{\partial x} + \frac{\partial j_2}{\partial y} + \frac{\partial j_3}{\partial z}\right) dx\, dy\, dz,$$

since on partial integration the last two terms vanish. Hence by (3.5)

$$\iiint j_1\, dx\, dy\, dz = \iiint x\frac{\partial \rho}{\partial t}\, dx\, dy\, dz = i2\pi\nu \iiint x\rho\, dx\, dy\, dz.$$

Thus it will suffice to discuss the electric moment.

When our proper functions are written as the tesseral spherical harmonics, the three appropriate types of moment are $x + iy$, z, $x - iy$. For determining what combinations occur we require the following easily proved relations :—

$$\iint \mathrm{P}_k^u \mathrm{P}_{k-1}^{u-1*} \sin\theta e^{-i\phi} \,.\, \sin\theta \, d\theta \, d\phi = -\iint \mathrm{P}_k^u \mathrm{P}_{k-1}^{u+1*} \sin\theta e^{i\phi} \,.\, \sin\theta \, d\theta \, d\phi$$

$$= \iint \mathrm{P}_k^u \mathrm{P}_{k-1}^{u*} \cos\theta \,.\, \sin\theta \, d\theta \, d\phi = \frac{4\pi}{(2k+1)(2k-1)} (k+u)!\,(k-u)! \tag{9.1}$$

For all other products of these types the integral vanishes. We may here note also the normalising relation

$$\iint \mathrm{P}_k^u \mathrm{P}_k^{u*} \sin\theta \, d\theta \, d\phi = \frac{4\pi}{2k+1} (k+u+1)!\,(k-u)! \tag{9.2}$$

With the help of (9.1) we can see what combinations might occur. It will evidently suffice to treat of only one type of polarisation and we shall take that corresponding to z. The following scheme then shows the only solutions which might, according to (9.1), combine with the first. We only mark the harmonic coefficients.

k	j	ψ_1	ψ_2	ψ_3	ψ_4
k	$k+\frac{1}{2}$	P_{k+1}^u	P_{k+1}^{u+1}	P_k^u	P_k^{u+1}
$k-1$	$k-\frac{1}{2}$	P_k^u	P_k^{u+1}	P_{k-1}^u	P_{k-1}^{u+1}
$k+1$	$k+\frac{1}{2}$	P_k^u	P_k^{u+1}	P_{k+1}^u	P_{k+1}^{u+1}
$k+1$	$k+\frac{3}{2}$	P_{k+2}^u	P_{k+2}^{u+1}	P_{k+1}^u	P_{k+1}^{u+1}
$k+3$	$k+\frac{5}{2}$	P_{k+2}^u	P_{k+2}^{u+1}	P_{k+3}^u	P_{k+3}^{u+1}
$k-1$	$k-\frac{3}{2}$	P_{k-2}^u	P_{k-2}^{u+1}	P_{k-1}^u	P_{k-1}^{u+1}

When we examine these, actually putting in the coefficients, we find that the last two vanish identically. This verifies the j rule, that only $j \to j \pm 1$ or $j \to j$ are possible combinations.† A similar result follows, of course, if we examine the other polarisations or the combinations of the second type of solution. We shall not give the numerical values here as these are well known, but one more point deserves remark. The radial integrations are, speaking

† Of course the k rule is required as well, for there are levels k, $k-\frac{1}{2}$ and $k+2$, $k+\frac{3}{2}$ which do not combine with k, $k+\frac{1}{2}$. Dirac's use of negative j does make the *statement* simpler.

accurately, different for the three lines arising from a given $k \to (k-1)$. They are, in fact,

$$\int (F_k F_{k-1} + G_k G_{k-1})\, r^3\, dr, \qquad \int (F_{-k-1} F_{-k} + G_{-k-1} G_{-k})\, r^3\, dr$$

and

$$\int (F_{-k-1} F_{k-1} + G_{-k-1} G_{k-1})\, r^3\, dr.$$

But to a first approximation we saw that F could be neglected beside G, while $G_{-k-1} = G_k$, so that to this approximation the three radial integrations are the same. This explains why the intensities bear simple numerical ratios to one another in doublet spectra.

10. When a uniform magnetic field is imposed on a doublet atom it is not possible to get an accurate solution, and we have to fall back on the method of perturbations. The simplest way of working out the Zeeman effect is to use the approximation of § 4, which reduces it to the work done in my earlier paper. That this is a sufficient approximation may be seen from the fact that it gives the doublet fine structure and treats the magnetic structure as of the same order of magnitude, which is just the degree of accuracy required to explain the observed effects. But a direct attack, starting from the accurate solution of (7.1) and superposing on it the magnetic field, is also interesting; it throws the solution into rather a different form because the levels of the fine structure are already separated, whereas in the earlier process they were attributed to a perturbation acting together with the magnetic field.

We must first see how the method of perturbations will go. The solutions of (2.2), when p_0 is replaced by W so that they give the levels, obey an orthogonal relation, as is easily proved directly; thus for any two solutions p, q either

$$W_p - W_q = 0, \quad \text{or} \quad \iiint \sum_{\lambda=1}^{4} \psi_\lambda^p \psi_\lambda^{q*}\, dx\, dy\, dz = 0. \tag{10.1}$$

In the case of degeneracy, where $W_p = W_q$, the partly arbitrary ψ_λ^q can be chosen so that (10.1) will still be true. This has already been done in our case. Let us suppose that on account of small changes in V and A the four equations (2.2) are affected by small extra terms which we may write as $P_1\psi \ldots P_4\psi$, each symbol signifying that any of the ψ's may enter into each equation. We require a solution near W_p and assume that it is of the form

$$\psi_\lambda = \Sigma\, a_s \psi_\lambda^s,$$

where the a_s's are small for all cases where W_s is not nearly equal to W_p. Notice the slight difference that we must make from the theory of perturbations of a degenerate system, on account of the fact that we have to allow for the interaction of solutions that initially are not quite coincident. Substituting in (2.2) we have

$$\sum_s \frac{1}{c}(W - W_s)\, a_s \psi_1^s + \sum_s a_s P_1 \psi^s = 0, \text{ etc.}$$

We multiply these by ψ_1^{q*}, etc., add the four equations together and integrate over space. We thus find

$$\frac{1}{c}(W - W_q)\, a_q \int \sum_{\lambda=1}^{4} \psi_\lambda^q \psi_\lambda^{q*}\, dx\, dy\, dz + \sum_s a_s \int \sum_{\lambda=1}^{4} \psi_\lambda^{q*} P_\lambda \psi^s\, dx\, dy\, dz = 0.$$

If there are a number of states q near p, even though not coincident with it, the associated a_q's need not be small and must be retained. We then form the determinant in the usual way, giving an algebraic equation for W, and afterwards we can determine the various ratios for a_q.

For a magnetic field along z we take $A_1 = -\frac{1}{2}Hy$, $A_2 = \frac{1}{2}Hx$ and so must add on to (2.2) terms

$$P_1\psi = -i\frac{eH}{2c}(x - iy)\,\psi_4, \quad P_2\psi = i\frac{eH}{2c}(x + iy)\,\psi_3,$$

$$P_3\psi = -i\frac{eH}{2c}(x - iy)\,\psi_2, \quad P_4\psi = i\frac{eH}{2c}(x + iy)\,\psi_1.$$

We hence get

$$\frac{1}{c}(W - W_q)\, a_q \int \sum_{\lambda=1}^{4} \psi_\lambda^q \psi_\lambda^{q*}\, dx\, dy\, dz + \frac{eH}{2c}\sum_s a_s [q;\, s] = 0, \qquad (10.2)$$

where

$$[q;\, s] = \int \{-i\psi_1^{q*}(x - iy)\,\psi_4^s + i\psi_2^{q*}(x + iy)\,\psi_3^s - i\psi_3^{q*}(x - iy)\,\psi_2^s + i\psi_4^{q*}(x + iy)\,\psi_1^s\}\, dx\, dy\, dz. \qquad (10.3)$$

This expression determines whether two levels q, s can interfere with one another in producing the Zeeman effect. If we apply the formulæ of (9.1) to any of the solutions (7.4) or (7.7), we at once see that unless the number u is the same for both q and s, the integral must vanish. Hence only levels with the same quantum number $m\,(= u + \frac{1}{2})$ can interfere. Further, we can build a table, of the same kind as was made in § 9 for combinations, to show what possible k, j values might give non-vanishing integrals, and, just as there, we find that some of the possibilities disappear on closer examination. We are left to consider interferences of $(k,\ k + \frac{1}{2})$ and $(k,\ k - \frac{1}{2})$ with themselves and one

another and also of $(k,\ k+\frac{1}{2})$ with $(k+2,\ k+\frac{3}{2})$ (the case $(k,\ k-\frac{1}{2})$ with $\left(k-2,\ k-\frac{s}{2}\right)$, which also occurs, is essentially the same as this last). The first set are what we are familiar with, but the last is rather unexpected. We shall prove that it is small of an order to be neglected, because it corresponds to the terms we should have by a second approximation.

Substituting out of (7.4), (7.7) and applying (9.1) we find

$$[k,\ k+\tfrac{1}{2};\ k+2,\ k+\tfrac{3}{2}] = \frac{4\pi}{2k+3}(k+u+2)\,!\,(k-u+1)\,!\int(\mathrm{F}_k\mathrm{G}_{k-3}+\mathrm{G}_k\mathrm{F}_{k-3})\,r^3\,dr.$$

In order to reduce the radial integration, we proceed as follows. Take the equations (7.5) and a similar pair in which k is replaced by another number l. Multiply the first of (7.5) by G_l, the second by F_l and add. This removes the terms in W and eV. Then interchange k and l and add the two expressions together. The result is

$$\frac{2\pi}{h}\cdot 2mc\,(\mathrm{F}_k\mathrm{G}_l+\mathrm{F}_l\mathrm{G}_k)+\frac{d}{dr}(\mathrm{F}_k\mathrm{F}_l+\mathrm{G}_k\mathrm{G}_l)-\frac{k+l}{r}\mathrm{G}_k\mathrm{G}_l+\frac{k+l+4}{r}\mathrm{F}_k\mathrm{F}_l=0.$$

Hence making a partial integration we find

$$\int(\mathrm{F}_k\mathrm{G}_l+\mathrm{F}_l\mathrm{G}_k)\,r^3\,dr=\frac{h}{4\pi mc}\int[\mathrm{G}_k\mathrm{G}_l\,(k+l+3)-\mathrm{F}_k\mathrm{F}_l\,(k+l+1)]\,r^2\,dr. \quad (10.4)$$

For the case $l=-k-3$, the term in G therefore vanishes. We know that F is smaller than G in a ratio of order $\gamma:1$; hence this integral bears to one where the coefficient of G does not vanish a ratio $\gamma^2:1$, which is the order of a term of the second, not the first approximation.

We are thus left with the result that only the two k levels interfere. Let us call $(k,\ k+\frac{1}{2})$, p and $(k,\ k-\frac{1}{2})$, q. Then our solution may be written as

$$\psi_\lambda = a\psi_\lambda^{\,p}+b\psi_\lambda^{\,q},$$

and we find by the application of (9.1) and (10.4) that

$$[p;\ p]=-4\pi\frac{2k+2}{2k+1}(2u+1)(k+u+1)!(k-u)\,!\,\frac{h}{4\pi mc}\int\left(\mathrm{G}_k{}^2-\frac{2k+1}{2k+3}\mathrm{F}_k{}^2\right)r^2dr,$$

$$[q;\ q]=-4\pi\frac{2k}{2k+1}(2u+1)(k+u)!(k-u-1)!\,\frac{h}{4\pi mc}\int\left(\mathrm{G}_{-k-1}-\frac{2k+1}{2k-1}\mathrm{F}_{-k-1}\right)r^2\,dr,$$

$$[p;\ q]=[q;\ p]=-\frac{4\pi}{2k+1}2\,(k+u+1)!\,(k-u)\,!\,\frac{h}{4\pi mc}\int\mathrm{G}_k\,\mathrm{G}_{-k-1}\,r^2\,dr.$$

We also require the normalisations. Using (9.2) these are

$$\int \sum_\lambda |\psi_\lambda^{\,p}|^2\, dx\, dy\, dz = 4\pi (k+u+1)!\,(k-u)! \int (F_k^2 + G_k^2)\, r^2\, dr,$$

$$\int \sum_\lambda |\psi_\lambda^{\,q}|^2\, dx\, dy\, dz = 4\pi (k+u)!\,(k-u-1)! \int (F_{-k-1}^2 + G_{-k-1}^2)\, r^2\, dr.$$

Now we have seen that in all these the terms in F_k, F_{-k-1} are small compared to those in G. Furthermore, we know that to the same approximation G_{-k-1} is proportional to G_k, and may be taken equal to it by a suitable definition. Hence all the radial integrations are the same, and they may therefore be omitted. The equation (10.2) thus becomes

$$\frac{a}{c}(W - W_p) - \frac{eH}{2c}\frac{h}{4\pi mc}\frac{2k+2}{2k+1}(2u+1)\,a - \frac{eH}{2c}\frac{h}{4\pi mc}\frac{2b}{2k+1} = 0.$$

$$\frac{b}{c}(W - W_q) - \frac{eH}{2c}\frac{h}{4\pi mc}\frac{2k}{2k+1}(2u+1)\,b - \frac{eH}{2c}\frac{h}{4\pi mc}\frac{(k+u+1)(k-u)}{2k+1}\,2a = 0.$$

To reduce this to the familiar form we write W_0 as the mean centre of the two lines, so that

$$W_p = W_0 + k\beta, \qquad W_q = W_0 - (k+1)\,\beta,$$

and $W = W_0 + \overline{W}$. Also take $\omega = \dfrac{eH}{2mc}\dfrac{h}{2\pi}$. Then we have

$$a\left\{\overline{W} - k\beta - \omega\frac{k+1}{2k+1}(2u+1)\right\} - b\omega\frac{1}{2k+1} = 0,$$

$$-a\omega\frac{(k+u+1)(k-u)}{2k+1} + b\left\{\overline{W} + (k+1)\,\beta - \omega\frac{k}{2k+1}(2u+1)\right\} = 0,$$

and from this we derive

$$\overline{W}^2 + \beta\overline{W} - k(k+1)\,\beta^2 + \omega(2u+1)(\overline{W} + \beta) + \omega^2 u(u+1) = 0,$$

which is the standard equation for the Zeeman effect in doublets. It will be seen that the algebra is a little more complicated than that which comes from forming the approximate equations in ψ_3, ψ_4, as was done before. We shall not work out intensities as nothing new would be found.

Summary.

Dirac's recent papers on the Quantum Theory of the Electron are discussed making use of the ordinary methods of differential equations instead of the non-commutative algebra used by him.

The equations are formed and proved invariant for relativistic transformations.

The emission of radiation from an atom containing Dirac's electron is discussed.

It is shown how Schrödinger's equation, and the pair of equations recently given by the present writer, are successive approximations to Dirac's.

A few points in the free motion of an electron are reviewed.

The equations are solved for the motion of an electron in a central field of force. They are shown to be expressible in terms of spherical harmonics and to lead to a doublet structure for the spectrum.

The discussion of the radial functions in the case of hydrogen is shown to lead exactly to Sommerfeld's original formula for the levels.

The rules of combination are considered in outline, and the Zeeman effect is worked out.